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Vanishing Theorems and Induced Representations

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Let G be a connected algebraic group over an algebraically closed field, and let B be a Borel subgroup of G. In this paper we show how I–III below can be reduced to proving that the first cohomology group of certain induced bundles on the projective line vanishes (see Section 4 for the precise statement).

I. If L is a line bundle on G/B which has a non-zero global section then $H^i(G/B, L) = 0$ for $i > 0$. Moreover, if X is any Schubert variety in G/B, then $H^i(X, L\mid_X) = 0$ for $i > 0$ and the natural map $H^0(G/B, L) \to H^0(X, L\mid_X)$ is surjective.

II. The Schubert varieties in G/B have rational singularities (hence are in particular normal and Cohen–Macaulay [14, p. 50]).

III. The formal character of $H^0(G/B, L)$, L as in I, is given by Demazure's character formula.

We prove the above mentioned vanishing theorem for the induced bundles on $\mathbf{P}^1$ only when G has semi-simple rank 2, but we hope to return to this problem and the related problem of constructing nice filtrations of induced modules in a later paper.

As a biproduct of our proof of the reduction theorems we obtain a necessary and sufficient condition under which a representation of B extends to G. This part of our work is related to (and influenced by) recent work of E. Cline, B. Parshall and L. Scott, [5, 6]. Their work [5] contains also another proof of a reduction theorem for the first part of I above.

In the characteristic zero case, M. Demazure proved I–III [8], and several partial results have been obtained in positive characteristic, notably [2, 12, 13]. The most general of these is [13] where a proof of I is given under the assumption that X is special (i.e. stable with respect to a certain maximal parabolic subgroup). See also [2].

The paper is organized as follows: In Section 1 we introduce notation and

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give some basic definitions. Section 2 contains results on the semi-simple rank 1 case. Demazure's construction of the "Bott–Samelson scheme" is recalled in Section 3 and used in Section 4 to prove the above mentioned reduction theorems. Section 5 deals with the groups of semi-simple rank 2 and finally in Section 6 we obtain the criteria for extensions of B-representations.

1. Preliminaries

k denotes an algebraically closed field which will serve as ground field for all schemes and groups throughout this paper.

1.1. *Induced bundles.* First let us point out that we use the words bundle, vector bundle and line bundle interchangeably with the words locally free sheaf, locally free sheaf of finite rank and invertible sheaf, respectively. Suppose X is a scheme on which an algebraic group H acts freely from the right in such a way that $p: X \to X/H$ exists and is a locally trivial fibration with group H. Let $\eta: H \to GL(E)$ be a representation of H on a vectorspace E. η induces a bundle $L_X(\eta)$ on X/H as follows: If U is an open subset of X/H then the sections of $L_X(\eta)$ over U are the morphisms $g: p^{-1}(U) \to E$ which satisfy $g(xh) = \eta(h)^{-1} g(x)$, $x \in p^{-1}(U)$, $h \in H$. Sometimes we will use the alternative notation $L_X(E)$ for this bundle and when no confusion can arise we omit the reference to X. In the following we shall in particular consider bundles on G/H induced in the above manner from representations of a closed subgroup H of a group G.

1.2. *Induced modules.* Let G be an algebraic group and H a closed subgroup of G. If E is a rational H-module then E induces a rational G-module $E\,|^G$ as follows: $E\,|^G =$ the set of morphisms $f: G \to E$ satisfying $f(xh) = h^{-1}f(x)$, $x \in G$, $h \in H$. The action of G on $E\,|^G$ is given by $g \cdot f(x) = f(g^{-1}x)$, $g, x \in G$, $f \in E\,|^G$. Comparing with 1.1 we see that $E\,|^G = H^0(G/H, L(E))$. If E is finite dimensional and G/H is a complete variety (i.e. H parabolic) then the induced module $E\,|^G$ is also finite dimensional (being a cohomology group of a coherent sheaf on a complete variety).

1.3. *Notation.* For use in the rest of this paper we will fix the following notation: G will be a connected algebraic group, and T (resp. B) a fixed maximal torus (resp. Borel subgroup containing T). We will always assume that G is reductive as the homogeneous space G/B does not change by passing to G modulo the unipotent radical. Hence we can speak of the set of roots R of G (w.r.t. T) and we will let the set of positive roots, R_+ be the roots of the opposite Borel subgroup B' of B. The roots of B are in other words the negative roots in our notation. S will denote the set of simple roots and if $\alpha \in R$ we let s_α (resp. $\alpha^\vee$) be the reflection (resp. the coroot) associated to α. We let W denote the Weyl group. An expression for $w \in W$ of the form $w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_n}$ with

$\beta_i \in S$ and n minimal is called a reduced expression for w and the length $\ell(w)$ of w is then equal to n. The element in W of maximal length is denoted w_0. The 1-dimensional unipotent subgroup corresponding to a root α is called U_α and we choose a homomorphism $\theta_\alpha: \mathbf{G}_a \to U_\alpha$ satisfying $t\theta_\alpha(z)\, t^{-1} = \theta_\alpha(\alpha(t)z)$, $t \in T$, $z \in k$. Finally we write $X(T)$ for the character group of T and $\mathbf{Z}[X(T)]$ for the group algebra of $X(T)$. When $\chi \in X(T)$ the corresponding element in $\mathbf{Z}[X(T)]$ is denoted e^χ.

1.4. *Formal characters.* Recall that if T acts on the finite dimensional vectorspace E then E can be decomposed, $E = \bigoplus_{\chi \in X(T)} E_\chi$ such that T acts via χ on E_χ. χ is called a weight on E if $E_\chi \neq 0$ and the dimension of E_χ is the multiplicity of χ. The formal character of E denoted $\operatorname{ch} E$ is by definition the following element in $\mathbf{Z}[X(T)]$

$$\operatorname{ch} E = \sum_{\chi \in X(T)} (\dim E_\chi) e^\chi.$$

Let us use this opportunity to introduce some operators on $\mathbf{Z}[X(T)]$ which will come up in connection with the formal characters of certain induced representations later on:

Suppose $\alpha \in R$. Following M. Demazure [9] we define a linear endomorphism Λ_α^0 of $\mathbf{Z}[X(T)]$ by

$$\Lambda_\alpha^0(u) = \frac{u - e^{-\alpha} s_\alpha(u)}{1 - e^{-\alpha}}.$$

It is immediately seen that

$$\Lambda_\alpha^0(e^\chi) = \begin{cases} e^\chi + e^{\chi+\alpha} + \cdots + e^{s_\alpha(\chi)} & \text{if } \langle \alpha^\vee, \chi \rangle \geqslant 0 \\ 0 \quad \text{if } \langle \alpha^\vee, \chi \rangle = -1 \\ -(e^{\chi+\alpha} + e^{\chi+2\alpha} + \cdots + e^{s_\alpha(\chi)-\alpha}) & \text{if } \langle \alpha^\vee, \chi \rangle \leqslant -2. \end{cases}$$

1.5. *Schubert varieties.* By the Schubert variety X_w in G/B associated to $w \in W$ we understand the image in G/B of the closure of the double cocell $BwB \subset G$. It is an immediate consequence of the Bruhat decomposition [3, IV.14] and [4, VI, Sect. 1, Corollary 2] that X_w is an $\ell(w)$-dimensional subvariety of G/B.

2. Induced Bundles on $\mathbf{P}^1$

It is very well-known that any vector bundle on the projective line can be decomposed into a sum of line bundles. Apparently this result can be traced all the way back to Dedekind and Weber [7] but in the more modern language it is due to Grothendieck [10].

In this section we assume that G has semi-simple rank 1 so that G/B is isomorphic to $\mathbf{P}^1$. Given a B-module E we can then ask: Which line bundles occur in a decomposition as above of the induced bundle $L(E)$? To answer this question we study the cohomology of $L(E)$. We will write $H^i(L(E))$ instead of $H^i(G/B, L(E)$ throughout this section and we will let $x_{-\alpha,m}$, $m \geqslant 0$ denote the standard basis of the enveloping algebra of the 1-dimensional Lie-algebra corresponding to the unipotent radical $U_{-\alpha}$ of B. Then $\theta_{-\alpha}(z)e = \sum_{m\geqslant 0} z^m x_{-\alpha,m}e$, $z \in k$, $e \in E$.

PROPOSITION 2.1. *Let E be a rational B-module and E_χ the weight space in E corresponding to $\chi \in X(T)$. Then $H^0(L(E)) \neq 0$ iff there exist $\chi \in X(T)$ with $\langle \alpha^\vee, \chi\rangle \geqslant 0$ and $e \in E_\chi$ such that $x_{-\alpha,m}e = 0$ for $m > \langle \alpha^\vee, \chi\rangle$.*

Proof. Assume first that $H^0(L(E)) \neq 0$ and choose a T-semi-invariant $h \in H^0(L(E))$ stable under $U_{-\alpha}$. This is possible as $H^0(L(E))$ must contain a B-stable line. Now G/B is pathed together of the two affine lines $U_\alpha B/B$ and $U_{-\alpha}s_\alpha B/B$ and by a good choice of θ_α and $\theta_{-\alpha}$ we may assume

$$\theta_\alpha(z) = \theta_{-\alpha}(z^{-1})\, s_\alpha t_\alpha(z)^{-1}\, \theta_{-\alpha}(-z^{-1}) \tag{2.2}$$

for all but finitely many $z \in k$. Here $t_\alpha(z) \in T$ has the property that $t_\alpha(z)e = z^{\langle \alpha^\vee, \chi\rangle}e$, $z \in k - \{0\}$ and $e \in E_\chi$. In fact we have in $SL_2(k)$ the relation

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-1} & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} -1 & 0 \\ -z^{-1} & -1 \end{pmatrix}, \quad z \in k - \{0\}$$

and there is a homomorphism $SL_2 \to G$ with central kernel [11, Proposition 4.2]. Applying h to (2.2) and using that $U_{-\alpha}h = h$ we get

$$h(\theta_\alpha(z)) = h(s_\alpha t_\alpha(z)^{-1}\, \theta_{-\alpha}(-z^{-1})) = \theta_{-\alpha}(z^{-1})\, t_\alpha(z)[h(s_\alpha)].$$

Let χ be the weight of $e = h(s_\alpha)$. Then the above equation can also be written

$$h(\theta_\alpha(z)) = \sum_{m\geqslant 0} z^{-m+\langle \alpha^\vee, \chi\rangle} x_{-\alpha,m}e$$

from which we conclude that $\langle \alpha^\vee, \chi\rangle \geqslant 0$ and $x_{-\alpha,m}e = 0$ for $m > \langle \alpha^\vee, \chi\rangle$ because the left hand side is a polynomial in z.

Assume next that $e \in E_\chi$ has the properties in the proposition. Define h_1: $U_{-\alpha}s_\alpha B \to E$ by $h_1(us_\alpha b) = b^{-1}e$, $u \in U_{-\alpha}$, $b \in B$ and h_2: $U_\alpha B \to E$ by $h_2(\theta_\alpha(z)b) = b^{-1}\sum_{m\geqslant 0} z^{-m+\langle \alpha^\vee, \chi\rangle} x_{-\alpha,m}e$, $z \in k$, $b \in B$. Then h_1 and h_2 are sections of $L(E)$ over $U_{-\alpha}s_\alpha B/B$ and $U_\alpha B/B$, respectively. Moreover, from (2.2) we see that they path together and hence give a global section of $L(E)$.

If $\chi \in X(T)$ we let $k(\chi)$ denote the 1-dimensional representation where the action of B is given via χ. Also if E is a B-module we write $E(\chi)$ short for $E \otimes k(\chi)$. Applying Proposition 2.1 to $E(\chi)$ we get

COROLLARY 2.3. *If E is a B-module and $\lambda \in X(T)$ then $H^0(L(E(\lambda))) \neq 0$ iff there exist $\chi \in X(T)$ with $\langle \alpha^\vee, \chi + \lambda \rangle \geqslant 0$ and $e \in E_\chi$ such that $x_{-\alpha,m}e = 0$ for $m > \langle \alpha^\vee, \chi + \lambda \rangle$.*

Suppose now $\delta \in X(T)$ has $\langle \alpha^\vee, \delta \rangle = 1$. Then $L(\delta)$ equals $O(1)$ when G/B is identified with $\mathbf{P}^1$. More generally $L(\chi)$ identifies with $O(\langle \alpha^\vee, \chi \rangle)$.

Suppose E is finite dimensional and let E^* denote the dual B-module (with the contragredient action). Then $L(E^*)$ is the dual sheaf of $L(E)$ and by Serre-duality we get $H^1(L(E)) \simeq H^0(L(E^*(-2\delta)))^*$. Applying Proposition 2.1 on $E^*(-2\delta)$ we obtain

COROLLARY 2.4. *Let E be a finite dimensional B-module. Then $H^1(L(E)) \neq 0$ iff there exist $\chi \in X(T)$ with $\langle \alpha^\vee, \chi \rangle \leqslant -2$ and $e \in E_\chi$ such that e is not in the image of $x_{-\alpha,m}\colon E \to E$ for $m > -\langle \alpha^\vee, \chi \rangle - 2$.*

For the rest of this section let us assume that E has finite dimension r. If $L(E) = \bigoplus_{i=1}^r L(n_i\delta)$ is a decomposition of $L(E)$ into line bundles we set $N(E) = \max\{n_i\}$ and $n(E) = \min\{n_i\}$. Recalling that $H^0(L(n\delta)) \neq 0$ iff $n \geqslant 0$ (and equivalently $H^1(L(n\delta)) \neq 0$ iff $n \leqslant -2$) we see that the numbers $N(E)$ and $n(E)$ also have the description

$$\begin{aligned} N(E) &= \max\{N \mid H^0(L(E(-N\delta))) \neq 0\}, \\ n(E) &= \max\{n \mid H^1(L(E(-(n+1)\delta)) = 0\}. \end{aligned}$$

Comparing this with Proposition 2.1 and Corollary 2.4 we get a description of $N(E)$ and $n(E)$ in terms of the B-module structure of E. We leave the precise formulation of this characterization to the reader and turn instead to the problem of finding the other n_i's. Here we shall use the exact sequences

$$\begin{aligned} 0 \longrightarrow H^0(L(E(-(n+1)\delta))) \xrightarrow{R_n} H^0(L(E(-n\delta))) \xrightarrow{V_n} E(-n\delta) \\ \longrightarrow H^1(L(E(-(n+1)\delta))) \longrightarrow H^1(L(E(-n\delta))) \longrightarrow 0 \end{aligned} \tag{2.5}$$

arising from the short exact sequence

$$0 \to L(-\delta) \to 0_{G/B} \to 0_1 \to 0$$

tensored by $L(E(-n\delta))$. Here V_n is evaluation at 1 and if $r \in H^0(L(\delta))$ is a B-semi-invariant then R_n may be identified with multiplication by r.

Using (2.5) we can now make the following useful observations

(2.6) If E is the restriction of a G-module then $n(E) = N(E) = 0$ (V_0 is an isomorphism).

(2.7) If F is a B-submodule of E then $N(E) \leqslant \max\{N(F)\, N(E/F)\}$ and $n(E) \geqslant \min\{n(F), n(E/F)\}$ with equality if $N(F) \geqslant N(E/F)$, resp. $n(F) \geqslant n(E/F)$.

(2.8) If $n(F) \geqslant k > N(E/F)$ then $0 \to H^0(L(E(-(k+1)\delta)))(\delta - \alpha) \to H^0(L(E(-k\delta))) \to F(-k\delta) \to 0$ is an exact sequence of B-modules.

(2.9) Suppose $0 = F_n \subset F_{n-1} \subset \cdots \subset F_0 = E$ is a B-filtration of E such that $F_k/F_{k+1} = C_k(\chi_k)$ with C_k a G-module and $\chi_k \in X(T)$. If $\langle \alpha^\vee, \chi_k \rangle \geqslant \langle \alpha^\vee, \chi_{k-1} \rangle$ for all k then $L(E) \simeq \oplus_k L(\langle \alpha^\vee, \chi_k \rangle \delta)$, where the multiplicity of $L(\langle \alpha^\vee, \chi_k \rangle \delta)$ equals $\dim C_k$.

A filtration of E as in (2.9) is called canonical if $\langle \alpha^\vee, \chi_k \rangle > \langle \alpha^\vee, \chi_{k-1} \rangle$ for all k. It is not hard to see that a canonical filtration exists. Indeed, we can take $F_k = (\operatorname{Im} V_k)(k\delta)$, omitting the k's where $\operatorname{Im} V_k = \operatorname{Im} V_{k-1}$. For more on canonical filtrations we refer to [6, Section 3].

We finish this section with a remark on the formal character of the cohomology of $L(E)$. If $\chi \in X(T)$ and $\langle \alpha^\vee, \chi \rangle \geqslant 0$ then the weights in $H^0(L(\chi))$ are χ, $\chi - \alpha, \ldots, s_\alpha(\chi)$. Dually if $\langle \alpha^\vee, \chi \rangle \leqslant -2$ then the weights in $H^1(L(\chi))$ are $\chi + \alpha, \chi + 2\alpha, \ldots, s_\alpha(\chi) - \alpha$, while both $H^0(L(\chi))$ and $H^1(L(\chi))$ vanish if $\langle \alpha^\vee, \chi \rangle = -1$. In the terminology of 1.4 we can therefore write

$$\operatorname{ch} H^0(L(\chi)) - \operatorname{ch} H^1(L(\chi)) = \Lambda_\alpha{}^0(e^\chi), \qquad \chi \in X(T).$$

As the alternating sum of the formal characters of the cohomology groups is additive we get

$$\operatorname{ch} H^0(L(E)) - \operatorname{ch} H^1(L(E)) = \Lambda_\alpha{}^0(\operatorname{ch} E). \tag{2.10}$$

3. The Demazure Desingularization

In this section we recall M. Demazure's construction of the "Bott–Samelson scheme" [8]. We also prove an easy lemma which will play an important role in the following sections.

Fix a reduced expression for w_0, $w_0 = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_N}$, $\beta_i \in S$, and set $\alpha_i = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_{i-1}}(\beta_i)$, $i = 1, 2, \ldots, N$. Then $\{\alpha_1, \alpha_2, \ldots, \alpha_N\} = R_+$ [4, VI.1, Corollary 2]. Let w_i be the reflection w.r.t. α_i, i.e. $w_i = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_{i-1}} s_{\beta_i} s_{\beta_{i-1}} \cdots s_{\beta_1}$ and define

$$\begin{aligned}
R_0 &= -R_+ = \{-\alpha_1, -\alpha_2, \ldots, -\alpha_N\} \\
R_1 &= w_1(R_0) = \{\alpha_1, -\alpha_2, \ldots, -\alpha_N\} \\
&\ \vdots \\
R_i &= w_i(R_{i-1}) = \{\alpha_1, \alpha_2, \ldots, \alpha_i, -\alpha_{i+1}, \ldots, -\alpha_N\} \\
&\ \vdots \\
R_N &= w_N(R_{N-1}) = \{\alpha_1, \alpha_2, \ldots, \alpha_N\} = R_+.
\end{aligned}$$

Let B_i be the Borel subgroup corresponding to R_i (so in particular $B_0 = B$ and $B_N = B'$), and let P_i be the parabolic subgroup corresponding to $R_{i-1} \cup R_i$. Set $X_0 = B_0$ and construct schemes X_i, $i = 1, 2, \ldots, N$ inductively by setting $X_i = X_{i-1} \times P_i/\sim$, where $\sim$ is the equivalence relation $(x, p) \sim (xb, b^{-1}p)$, $x \in X_{i-1}$, $p \in P_i$, $b \in B_{i-1}$ and where the action of B_i on X_i comes from right multiplication on P_i. Set $Y_i = X_i/B_i$ and let $p_i\colon X_i \to Y_i$ be the canonical projection and $f_i\colon Y_i \to Y_{i-1}$ be the morphism which takes the class of (x, p), $x \in X_{i-1}$, $p \in P_i$ into $p_{i-1}(x)$. We then have the diagram

$$\begin{tikzcd}
X_0 \arrow[d, "p_0"] & X_1 \arrow[d, "p_1"] & & X_{i-1} \arrow[d, "p_{i-1}"] & X_i \arrow[d, "p_i"] & & X_N \arrow[d, "p_N"] \\
Y_0 \arrow[r, shift left, "\sigma_1"] & Y_1 \arrow[l, shift left, "f_1"] \arrow[r, shift left] & \cdots \arrow[l, shift left] & Y_{i-1} \arrow[r, shift left, "\sigma_i"] & Y_i \arrow[l, shift left, "f_i"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left, "\sigma_N"] & Y_N . \arrow[l, shift left, "f_N"]
\end{tikzcd}$$

Here σ_i is the section of f_i which takes $p_{i-1}(x)$, $x \in X_{i-1}$ into the class of (x, w_i). We see that Y_i is a smooth projective variety and that f_i is a locally trivial fibration with fiber $P_i/B_i = \mathbf{P}^1$. Moreover, the multiplication map $P_1 \times P_2 \times \cdots \times P_N \to G$ induces a morphism $Y_N \to G/B'$. This is the so called "Bott–Samelson morphism" [8, 3.8]. Composing it with the natural isomorphism $G/B' \to G/B$ ($gB' \mapsto gw_0B$, $g \in G$) gives a morphism $\Phi\colon Y_N \to G/B$ with the properties

(3.1) Φ is birational [8, 3.8].

(3.2) $\Phi_i = \Phi \circ \sigma_N \circ \cdots \circ \sigma_{i+1}$ is a birational morphism from Y_i to the Schubert variety in G/B associated with $s_{\beta_1} s_{\beta_2} \cdots s_{\beta_i} \in W$ [8, 3.11].

In addition to these properties of the desingularization sequence we shall need the following.

Lemma 3.3. *Let $1 \leqslant i \leqslant N$ and let $\eta\colon B_i \to GL(E)$ be a representation of B_i on the vectorspace E. Then*

$$f_{i*} L_{X_i}(\eta) \cong L_{X_{i-1}}(\eta')$$

where $\eta'\colon B_{i-1} \to GL(E')$ is the restriction to B_{i-1} of the natural action of P_i on $E' = H^0(P_i/B_i, L(E))$.

Proof. Let U be an open subset of Y_{i-1}. By definition the sets of sections over U of the two sheaves in question are

$$\Gamma_1 = \{g\colon p_i^{-1} f_i^{-1}(U) \to E \mid g(xb) = \eta(b)^{-1} g(x),\ x \in p_i^{-1} f_i^{-1}(U),\ b \in B_i\}$$

and

$$\Gamma_2 = \{h\colon p_{i-1}^{-1}(U) \to E' \mid h(xb) = \eta'(b)^{-1} h(x),\ x \in p_{i-1}^{-1}(U),\ b \in B_{i-1}\}.$$

Define $f_U{}^1\colon \Gamma_1 \to \Gamma_2$ by

$$f_U{}^1(g)(x)\colon p \mapsto g(\overline{(x, p)}),\ g \in \Gamma_1\,,\ x \in p_{i-1}^{-1}(U),\ p \in P_i$$

and $f_U{}^2\colon \Gamma_2 \to \Gamma_1$ by

$$f_U{}^2(h)\colon \overline{(x, p)} \mapsto h(x)(p),\ h \in \Gamma_2\,,\ \overline{(x, p)} \in p_i^{-1} f_i^{-1}(U).$$

Let us check that $f_U{}^2$ is well-defined. First we note that if $\overline{(x, p)} \in p_i^{-1} f_i^{-1}(U)$ then $x \in p_{i-1}^{-1}(U)$ so that we can indeed apply $h \in \Gamma_2$ to x. Next we have to verify that if $\overline{(x, p)} = \overline{(x', p')}$ then $h(x)(p) = h(x')(p')$. But $\overline{(x, p)} = \overline{(x', p')}$ means that $x' = xb$ and $p' = b^{-1}p$ for some $b \in B_{i-1}$. For $h \in \Gamma_2$ we have $h(xb)(b^{-1}p) = [\eta'(b)^{-1}\, h(x)](b^{-1}p) = h(x)(p)$ where the last equality follows from the definition of η'. Finally we must show that $\overline{(x, p)} \mapsto h(x)(p)$ belongs to Γ_1, i.e. that $h(x)(pb) = \eta(b)^{-1}[h(x)(p)]$. This follows from the fact that $h(x) \in E'$.

Similar elementary calculations show that $f_U{}^1$ is well-defined and it is also easy to see that $f_U{}^2 = (f_U{}^1)^{-1}$. The collection $(f_U{}^1)$, U open in Y_{i-1}, gives therefore an isomorphism of sheaves $f_{i*}L_{X_i}(\eta) \to L_{X_{i-1}}(\eta')$.

4. Reduction Theorems

The following two propositions are easy generalizations of the results in [8, 5.1]. They will enable us to compare the cohomology of coherent sheaves on G/B with the cohomology of corresponding sheaves on the "Bott–Samelson scheme" introduced in the previous section.

Proposition 4.1. *Let $f\colon X \to Y$ be a morphism between two proper schemes over a noetherian ring, let F be a coherent sheaf on X and L an ample line bundle on Y. If $H^p(X, F \otimes f^*L^n) = 0$ for $p > 0$ and n large, then $R^pf_*F = 0$ for $p > 0$.*

Proof. As R^qf_*F are coherent for all q and L is ample we have $H^p(Y, R^qf_*F \otimes L^n) = 0$ for $p > 0$, $q \geqslant 0$ and n large. By the projection formula $R^qf_*F \otimes L^n = R^qf_*(F \otimes f^*L^n)$ and the above vanishing implies therefore that the Leray spectral sequence $H^p(Y, R^qf_*(F \otimes f^*L^n)) \Rightarrow H^{p+q}(X, F \otimes f^*L^n)$ degenerates when n is large. Hence $H^0(Y, R^qf_*F \otimes L^n) = H^q(X, F \otimes f^*L^n)$ for all q and $n \gg 0$. The ampleness of L and the assumption that $H^q(X, F \otimes f^*L^n) = 0$ for $p > 0$ and n large force R^qf_*F to be zero for $q > 0$.

Proposition 4.2. *Let f be as above and let $X' \subset X$ be a closed subscheme with image Y' in Y. Suppose L is an ample line bundle on Y such that the canonical map $H^0(X, f^*L^n) \to H^0(X', f^*L^n)$ is surjective for n large. If $f_*0_X = 0_Y$ then $f_*0_{X'} = 0_{Y'}$.*

Proof. Via the projection formula we have $H^0(Y, f_*0_X \otimes L_n) = H^0(X, f^*L^n)$ and $H^0(Y', f_*0_{X'} \otimes L^n) = H^0(X', f^*L^n)$. Hence $H^0(Y, f_*0_X \otimes L^n) \to H^0(Y', f_*0_{X'} \otimes L^n)$ is surjective and as L is ample this implies that $f_*0_X \to f_*0_{X'}$ is an epimorphism. If therefore $f_*0_X = 0_Y$ we have that $0_Y \to f_*0_{X'}$ is surjective and the proposition follows.

Let $\chi \in X(T)$ and set $\chi_i = w_{i+1}w_{i+2} \cdots w_N w_0(\chi)$. Set $E^\chi_{ii} = \bar{E}^\chi_{ii} = k(\chi_i)$ and define E^χ_{ij} (resp. $\bar{E}^\chi_{ij}$) for $j < i$ by $E^\chi_{ij} = H^0(P_{j+1}/B_{j+1}, L(E^\chi_{ij+1}))$ (resp. $\bar{E}^\chi_{ij} = H^0(P_{j+1}/B_{j+1}, L(\bar{E}^\chi_{ij+1}(\alpha_{j+1}))))$. In the following we will make the following assumptions:

(*) If $\langle \alpha^\vee, \chi \rangle \geqslant -1$ for all $\alpha \in S$ then $H^1(P_j/B_j, L(E^\chi_{ij})) = 0$ for all $j \leqslant i$.

(**) If $\langle \alpha^\vee, \chi \rangle \geqslant 0$ for all $\alpha \in S$ then the canonical map $E^\chi_{i0} \to E^\chi_{i-10}$ is surjective.

(***) If $\langle \alpha^\vee, \chi \rangle < 0$ for all $\alpha \in S$ then $H^1(P_j/B_j, L(\bar{E}^\chi_{ij}(\alpha_j))) = 0$ for all $j \leqslant i$.

Here are some consequences of these assumptions:

THEOREM 4.3. *Let* $\chi \in X(T)$.

(i) *If* $\langle \alpha^\vee, \chi \rangle \geqslant -1$ *for all* $\alpha \in S$ *then* $H^p(Y_i, L_{X_i}(\chi_i)) = 0$ *for* $p > 0$, $i = 1, 2, \ldots, N$.

(ii) *If* $\langle \alpha^\vee, \chi \rangle \geqslant 0$ *for all* $\alpha \in S$ *then the canonical map* $H^0(Y_i, L_{X_i}(\chi_i)) \to H^0(Y_{i-1}, L_{X_{i-1}}(\chi_{i-1}))$ *is surjective.*

(iii) *If* $\langle \alpha^\vee, \chi \rangle > 0$ *then* $H^p(Y_i, \Omega^i_{Y_i} \otimes L_{X_i}(\chi_i)) = 0$ *for* $p > 0$.

Proof. (i) It is obviously enough to prove that $R^1 f_{j*}(f_{j+1*} \cdots f_{i*} L_{X_i}(\chi_i)) = 0$ for $j = i, i-1, \ldots, 1$. Via base change this reduces to showing that $H^1(P_j/B_j, f_{j+1*} \cdots f_{i*} L_{X_i}(\chi_i)) = 0$, $j = i, i-1, \ldots, 1$ and repeated use of Lemma 3.3 shows that $f_{j+1*} \cdots f_{i*} L_{X_i}(\chi_i) = L(E^\chi_{ij})$ in the above notation. Hence (i) follows from (*).

(ii) Lemma 3.3 shows that $H^0(Y_i, L_{X_i}(\chi_i)) = E^\chi_{i0}$ and $H^0(Y_{i-1}, L_{X_{i-1}}(\chi_{i-1})) = E^\chi_{i-10}$. Conclusion by (**).

(iii) The sheaf of differentials relative to f_i is equal to $L_{X_i}(\alpha_i)$ [8, 2.5]. The short exact sequence $0 \to f_i^* \Omega^1_{Y_{i-1}} \to \Omega^1_{Y_i} \to \Omega^1_{Y_i/Y_{i-1}} \to 0$ implies therefore that $\Omega^i_{Y_i} = f_i^* \Omega^{i-1}_{Y_{i-1}} \otimes L(\alpha_i)$. Hence $R^p f_{i*}(\Omega^i_{Y_i} \otimes L_{X_i}(\chi_i)) = \Omega^{i-1}_{Y_{i-1}} \otimes R^p f_{i*} L_{X_i}(\chi_i + \alpha_i)$, $p = 0, 1$. Using this together with Lemma 3.3 repeatedly as in the proof of (i), we see that (iii) follows if $H^1(P_j/B_j, L(\bar{E}^\chi_{ij}(\alpha_j))) = 0, j = i, i-1, \ldots, 1$, i.e. if (***) holds.

Observing that $\Phi_i^* L(\chi) = L_{X_i}(\chi_i)$ we get from this theorem combined with the Propositions 4.1 and 4.2

COROLLARY 4.4. *Set* $w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_i}$. $\Phi_i\colon Y_i \to X_w$ *is a rational resolution, i.e.* $\Phi_{i*}0_{Y_i} = 0_{X_w}$ *and* $R^p\Phi_{i*}0_{Y_i} = R^p\Phi_{i*}\Omega^i_{Y_i} = 0$ *for* $p > 0$.

Remark 4.5. Note that to obtain Corollary 4.4 we actually don't need the full force of the assumptions (*)–(***). The existence of a single strictly dominant weight χ such that the conclusions in (*)–(***) hold for $n\chi$, when n is large is enough.

From Corollary 4.4 follows in particular that the Leray spectral sequence $H^p(X_w, R^q\Phi_{i*}(\Phi_i^*F)) \Rightarrow H^{p+q}(Y_i, \Phi_i^*F)$, F a coherent sheaf on X_w, degenerates and we obtain $H^p(X_w, F) = H^p(Y_i, \Phi_i^*F)$. Taking $F = L(\chi)$ we get via Theorem 4.3

THEOREM 4.6. *Let* $\chi \in X(T)$ *and* $w \in W$.

(i) *If* $\langle\alpha^\vee, \chi\rangle \geqslant -1$ *for all* $\alpha \in S$ *then* $H^p(G/B, L(\chi)) = H^p(X_w, L(\chi)) = 0$ *for* $p > 0$.

(ii) *If* $\langle\alpha^\vee, \chi\rangle \geqslant 0$ *for all* $\alpha \in S$ *then the restriction map* $H^0(G/B, L(\chi)) \to H^0(X_w, L(\chi))$ *is surjective.*

(iii) *If* $\langle\alpha^\vee, \chi\rangle < 0$ *for all* $\alpha \in S$ *then* $H^p(X_w, L(\chi)) = 0$ *for* $p < \ell(w)$.

Let $\chi \in X(T)$ be dominant. The vanishing of $H^p(G/B, L(\chi))$ for $p > 0$ and the fact that G/B is defined over $\mathbf{Z}$ implies that there exists a finitely generated free $\mathbf{Z}$-module $V_{\mathbf{Z}}(\chi)$ such that $H^0(G/B, L(\chi)) = V_{\mathbf{Z}}(\chi) \otimes k$. When the characteristic of k is zero the complete reducibility theorem together with the obvious fact that $H^0(G/B, L(\chi))$ has a unique B-stable line imply that $H^0(G/B, L(\chi))$ is irreducible. The above enables us to compute the formal character of $H^0(G/B, L(\chi))$ (in any characteristic).

THEOREM 4.7 (Demazure's character formula [9]). *Let* $\chi \in X(T)$ *and suppose* $\langle\alpha^\vee, \chi\rangle \geqslant 0$. *Then*

$$\operatorname{ch} H^0(G/B, L(\chi)) = \Lambda^0_{\alpha_1}\Lambda^0_{\alpha_2} \cdots \Lambda^0_{\alpha_N}(e^\chi).$$

Proof. We have $H^0(G/B, L(\chi)) = E^\chi_{N0}$. Combining (2.10) and (*) we get $\operatorname{ch} E^\chi_{N0} = \Lambda^0_{-\alpha_1}(\operatorname{ch} E^\chi_{N_1}) = \cdots = \Lambda^0_{-\alpha_1}\Lambda^0_{-\alpha_2} \cdots \Lambda^0_{-\alpha_N}(e^{w_0(\chi)}) = \Lambda^0_{-w_0(\alpha_1)}\Lambda^0_{-w_0(\alpha_2)} \cdots \Lambda^0_{-w_0(\alpha_N)}(e^\chi) = \Lambda^0_{\alpha_1}\Lambda^0_{\alpha_2} \cdots \Lambda^0_{\alpha_N}(e^\chi)$.

5. SEMI-SIMPLE RANK 2

In this section we will assume that G has semi-simple rank 2 and we will prove that then the assumptions (*)–(***) in the previous section hold for G.

With notation as before let E be a B_i-module. Suppose E has a B_i-filtration $F_i^\cdot\colon 0 = F_i^r \subset F_i^{r-1} \subset \cdots \subset F_i^0 = E$ with the properties

(a) $F_i^k/F_i^{k+1} = C_i^k(\chi_i^k)$ where C_i^k is a P_i-module and $\chi_i^k \in X(T)$.

(b) $w_i(F_i^{p_i^k(F_i^\cdot)}) = C_{ii-1}^k(\chi_{ii-1}^k)$ as B_{i-1}-modules where C_{ii-1}^k is a P_{i-1}-module, $\chi_{ii-1}^k \in X(T)$ and $p_i^k(F_i^\cdot) = \min\{p \mid \langle -\alpha_i^\vee, \chi_i^p\rangle \geqslant k\}$.

Set $n_i^k(F_i^\cdot) = \langle -\alpha_i^\vee, \chi_i^k\rangle$, $n_{i-1}^k(F_i^\cdot) = \langle -\alpha_{i-1}^\vee, \chi_{ii-1}^k - k\alpha_i\rangle$ and then inductively

$$n_j^k(F_i^\cdot) = -p_{j+1}^k(F_i^\cdot) - k\langle \beta_j^\vee, \beta_{j+1}\rangle, \qquad j < i-1$$

$$p_j^k(F_i^\cdot) = \min\{p \mid n_j^p(F_i^\cdot) \geqslant k\}, \qquad j \leqslant i-1.$$

We will say that $F_i^\cdot$ is a *good* (respectively *nice*) *filtration* if it also has the property

(c) $n_j^k(F_i^\cdot) \geqslant n_j^{k-1}(F_i^\cdot) \geqslant -1$ (respectively 0) for all k, j.

PROPOSITION 5.1. *If the B_i-module E has a good filtration then so has the B_{i-1}-module $H^0(P_i/B_i, L(E))$.*

Proof. Let $F_i^\cdot: 0 = F_i^r \subset F_i^{r-1} \subset \cdots \subset F_i^0 = E$ be a good filtration of E. Set $\delta_i = s_{\beta_1}s_{\beta_2}\cdots s_{\beta_{i-1}}(\omega_i)$ where ω_i is the fundamental weight with respect to β_i. We define then a B_{i-1}-filtration $F_{i-1}^\cdot$ of $H^0(P_i/B_i, L(E))$ by setting

$$F_{i-1}^k = H^0(P_i/B_i, L(E(k\delta_i))(-k\delta_i), \quad k = 0, 1, \ldots, n_i^{r-1}(F_i^\cdot).$$

The inclusion of F_{i-1}^k in F_i^{k-1} is given by multiplication by a B_{i-1}-semi-invariant $r \in H^0(P_i/B_i, L(\delta_i))$ (note that $\langle \alpha_i^\vee, \delta_i\rangle = \langle \beta_i^\vee, \omega_i\rangle = 1$ and that r has weight δ_i), and the evaluation at w_i-map from $H^0(P_i/B_i, L(E(k\delta_i)))$ into $w_i(E(k\delta_i))$ gives a B_{i-1}-map from F_{i-1}^k into $w_i(E)(-k\alpha_i)$, see 2.5. From 2.8 and 2.9 follows

$$F_{i-1}^k/F_{i-1}^{k+1} = w_i(F_i^{p_i^k(F_i^\cdot)})(-k\alpha_i) \qquad \text{as} \quad B_{i-1}\text{-modules}. \tag{5.2}$$

As $F_i^\cdot$ has property (b) we see from (5.2) that $F_{i-1}^\cdot$ has property (a) with $\chi_{i-1}^k = \chi_{ii-1}^k - k\alpha_i$. Hence $n_{i-1}^k(F_{i-1}^\cdot) = \langle -\alpha_{i-1}^\vee, \chi_{i-1}^k\rangle = \langle -\alpha_{i-1}^\vee, \chi_{ii-1}^k - k\alpha_i\rangle = n_{i-1}^k(F_i^\cdot)$. Observe now that if F is a P_i-module then $w_{i-1}(F)$ is a P_{i-2}-module (as $\beta_{i-2} = \beta_i$). From this observation and the definition of F_{i-1}^k we see that $F_{i-1}^\cdot$ has property (b) and that $\chi_{(i-1)(i-2)}^k = -p_{i-1}^k(F_{i-1}^\cdot)\, w_{i-1}(\delta_i)$. We saw above that $n_{i-1}^k(F_{i-1}^\cdot) = n_{i-1}^k(F_i^\cdot)$. Hence $p_{i-1}^k(F_{i-1}^\cdot) = p_{i-1}^k(F_i^\cdot)$ and we also get $n_{i-2}^k(F_{i-1}^\cdot) = \langle -\alpha_{i-2}^\vee, -p_{i-1}^k(F_i^\cdot)\, w_{i-1}(\delta_i) - k\alpha_{i-1}\rangle = -p_{i-1}^k(F_i^\cdot) - k\langle \beta_{i-2}^\vee, \beta_{i-1}\rangle = n_{i-2}^k(F_i^\cdot)$. As the numbers $n_j^k(F_{i-1}^\cdot)$ for $j < i-2$ are defined inductively from $n_{i-1}^k(F_{i-1}^\cdot)$ and $n_{i-2}^k(F_{i-1}^\cdot)$ we conclude that $n_j^k(F_{i-1}^\cdot) = n_j^k(F_i^\cdot)$ for all $j \leqslant i-1$. Whence $F_{i-1}^\cdot$ has property (c).

When E is a B_i-module we define the B_j-modules E_{ij}, $j \leqslant i$, as follows. We set $E_{ii} = E$ and then $E_{ij} = H^0(P_{j+1}/B_{j+1}, L(E_{ij+1}))$, $j < i$.

COROLLARY 5.2. (i) *If E has a good filtration then* $H^1(P_j/B_j, L(E_{ij})) = 0$, $j = i, i-1, \ldots, 1$.

(ii) *If E has a nice filtration then the natural map* $E_{i0} \to [w_i(E)]_{i-10}$ *is surjective.*

Proof. (i) is an immediate consequence of Proposition 5.1 and 2.9. To prove (ii) set $E' = H^0(P_i/B_i, L(E(\delta_i)))(-\delta_i)$. Then E' is the kernel of the natural map $E_{ii-1} \to w_i(E)$ and we see that to prove (ii) we have to show that $H^1(P_j/B_j, L(E'_{i-1j})) = 0$, $j = i-1, i-2, \ldots, 1$. Set now $F'^k_{i-1} = H^0(P_i/B_i, L(E((k+1)\delta_i)))(-(k+1)\delta_i)$, and note that this filtration equals the filtration of $H^0(P_i/B_i, L(E))$ constructed in the proof of Proposition 5.1 shifted by 1. The same reasoning as used there shows that when E has a nice filtration then the filtration $F'^{\cdot}_{i-1}$ is good and we conclude as before via 2.9.

Let again $F_i^{\cdot}$ be a filtration of the B_i-module E and suppose $F_i^{\cdot}$ has property (a). Set $m_i^k(F_i^{\cdot}) = \langle -\alpha_i^{\vee}, \chi_i^k \rangle - 2$ and $q_i^k(F_i^{\cdot}) = \min\{q \mid m_i^q(F_i^{\cdot}) \geqslant k\}$. Suppose $F_i^{\cdot}$ satisfies (b) with the numbers $q_i^k(F_i^{\cdot})$ replacing $p_i^k(F_i^{\cdot})$ and set $m_{i-1}^k(F_i^{\cdot}) = \langle -\alpha_{i-1}^{\vee}, \chi_{ii-1}^k - (k+1)\alpha_i \rangle - 2$. For $j \leqslant i-2$ (respectively $j \leqslant i-1$) we set $m_j^k(F_i^{\cdot}) = -q_{j+1}^k(F_i^{\cdot}) - (k+1)\langle \beta_j^{\vee}, \beta_{j+1} \rangle - 2$ (respectively $q_j^k(F_i^{\cdot}) = \min\{q \mid m_j^q(F_i^{\cdot}) \geqslant k\}$). Call $F_i^{\cdot}$ a *very nice filtration* if it also has the property

$$\text{(c}'\text{)} \quad m_j^k(F_i^{\cdot}) \geqslant m_j^{k-1}(F_i^{\cdot}) \geqslant -1 \qquad \text{for all} \quad k, j.$$

Arguing as above we find that if E has a very nice filtration so has $H^0(P_i/B_i, L(E(\alpha_i)))$. Defining $\bar{E}_{ii} = E$ and $\bar{E}_{ij} = H^0(P_j/B_j, L(\bar{E}_{ij+1}(\alpha_j)))$ we get therefore via 2.9.

COROLLARY 5.3. *If E has a very nice filtration then* $H^1(P_j/B_j, L(\bar{E}_{ij}(\alpha_j))) = 0$, $j = i, i-1, \ldots, 1$.

Let now $\chi \in X(T)$ and set $\chi_i = w_{i+1}w_{i+2} \cdots w_N w_0(\chi)$. Let $F_i^{\cdot}: 0 = F_i^1 \subset F_i^0 = k(\chi_i)$ be the trivial filtration. The above two corollaries show that to prove (*)–(***) it is enough to verify

(i) If $\langle \alpha^{\vee}, \chi \rangle \geqslant -1$ for all $\alpha \in S$ then $F_i^{\cdot}$ is good.

(ii) If $\langle \alpha^{\vee}, \chi \rangle \geqslant 0$ for all $\alpha \in S$ then $F_i^{\cdot}$ is nice.

(iii) If $\langle \alpha^{\vee}, \chi \rangle > 0$ for all $\alpha \in S$ then $F_i^{\cdot}$ is very nice.

As $F_i^{\cdot}$ obviously satisfies (a) and (b) we have only left to check the inequalities (c) (resp. (c′)) for the numbers $n_j^k(F_i^{\cdot})$ (resp. $m_j^k(F_i^{\cdot})$). Below we have done this for the case where G is of type G_2 and $i = 6$ leaving the remaining (easier) cases to the reader.

Let α and β be the two simple roots for a root system of type G_2 with α short. Suppose $\beta_6 = \beta$(i.e. the reduced expression used for w_0 is $w_0 = (s_\alpha s_\beta)^3$). Set $r = \langle \alpha^{\vee}, \chi \rangle$ and $\langle \beta^{\vee}, \chi \rangle = s$. If q is a rational number we use the notation

$[q]_+ = \min(n \in \mathbf{Z} \mid n \geqslant q\}$ and $[q]_- = \max\{n \in \mathbf{Z} \mid n \leqslant q\}$. Omitting reference to $F_6^{\cdot}$ (the trivial filtration of $k(\chi_6)$) we get

$$n_6^0 = \langle -\alpha_6^\vee, \chi_6 \rangle = \langle \beta^\vee, \chi \rangle = s;\ m_6^0 = s - 2.$$

$$n_5^k = \langle -\alpha_5^\vee, \chi_5 - k\alpha_6 \rangle = r + k,\ k = 0, 1, \ldots, s;$$

$$m_5^k = \langle -\alpha_5^\vee, \chi_5 - (k+1)\alpha_6 \rangle - 2 = r + k - 1,\ k = 0, 1, \ldots, s-2;$$

$$p_5^k = \begin{cases} 0, & k \leqslant r \\ k - r, & r < k \leqslant r + s \end{cases};\quad q_5^k = \begin{cases} 0, & k \leqslant r - 1 \\ k + 1 - r, & r - 1 < k \leqslant r + s - 2 \end{cases}.$$

$$n_4^k = -p_5^k + 3k = \begin{cases} 3k, & k \leqslant r \\ 2k + r, & r < k \leqslant r + s \end{cases};$$

$$m_4^k = -q_5^k + 3(k+1) - 2 = \begin{cases} 3k + 1, & k \leqslant r - 1 \\ 2k + r, & r - 1 < k \leqslant r + s - 2 \end{cases};$$

$$p_4^k = \begin{cases} [\tfrac{1}{3}k]_+, & k \leqslant 3r \\ [\tfrac{1}{2}(k - r)]_+, & 3r < k \leqslant 3r + 2s \end{cases};$$

$$q_4^k = \begin{cases} [\tfrac{1}{3}(k - 1)]_+, & k \leqslant 3r - s \\ [\tfrac{1}{2}(k - r)]_+, & 3r - 2 < k \leqslant 3r + 2s - 4 \end{cases}.$$

$$n_3^k = -p_4^k + k = \begin{cases} [\tfrac{2}{3}k]_-, & k \leqslant 3r \\ [\tfrac{1}{2}(r + k)]_-, & 3r < k \leqslant 3r + 2s \end{cases};$$

$$m_3^k = -q_4^k + (k+1) - 2 = \begin{cases} [\tfrac{2}{3}(k - 1)]_-, & k \leqslant 3r - 2 \\ [\tfrac{1}{2}(k + r - 2)]_-, & 3r-2 < k \leqslant 3r + 2s-4 \end{cases};$$

$$p_3^k = \begin{cases} [\tfrac{3}{2}k]_+, & k \leqslant 2r \\ 2k - r, & 2r < k \leqslant 2r + s \end{cases};$$

$$q_3^k = \begin{cases} [\tfrac{1}{2}(3k + 2)]_+, & k \leqslant 2r - 2 \\ 2k - r + 2, & 2r - 2 < k < 2r + s - 3 \end{cases}.$$

$$n_2^k = -p_3^k + 3k = \begin{cases} [\tfrac{3}{2}k]_-, & k \leqslant 2r \\ k + r, & 2r < k \leqslant 2r + s \end{cases};$$

$$m_2^k = -q_3^k + 3(k+1) - 2 = \begin{cases} [\tfrac{3}{2}k]_-, & k \leqslant 2r - 2 \\ k + r - 1, & 2r - 2 < k \leqslant 2r + s - 3 \end{cases};$$

$$p_2^k = \begin{cases} [\tfrac{2}{3}k]_+, & k \leqslant 3r \\ k - r, & 3r < k \leqslant 3r + s \end{cases};$$

$$q_2^k = \begin{cases} [\tfrac{2}{3}k]_+, & k \leqslant 3r - 3 \\ k - r + 1, & 3r - 3 < k \leqslant 3r + s - 4 \end{cases}.$$

$$n_1^k = -p_2^k + k = \begin{cases} [\tfrac{1}{3}k]_-, & k \leqslant 3r \\ r, & 3r < k \leqslant 3r + s \end{cases};$$

$$m_1^k = -q_2^k + (k+1) - 2 = \begin{cases} [\tfrac{1}{3}(k - 3)]_-, & k \leqslant 3r - 3 \\ r - 2, & 3r - 3 < k \leqslant 3r + s - 4 \end{cases}.$$

We see that n_j^k and m_j^k satisfy the desired inequalities.

6. Extensions of Representations

The results in this section do not depend on the assumptions (*)–(***) made in Section 4. We first show how the main result of [5] can be proved using our setup:

THEOREM 6.1. *A representation of B extends to G iff it extends to every minimal parabolic subgroup.*

Proof. Let E be a B-module which extends to every minimal parabolic subgroup. In the notation of Section 3 we have $H^0(G/B, L(E)) = H^0(Y_N, L(w_0(E))) = H^0(Y_{N-1}, f_{N*}L(w_0(E))) = \cdots = H^0(Y_1, f_{2*}f_{3*} \cdots f_{N*}L(w_0(E)))$. The assumption on E implies that $H^0(P_j/B_j, L(w_{j+1}w_{j+2} \cdots w_N w_0(E))) = w_j w_{j+1} \cdots w_N w_0(E)$ as B_j-modules (see (2.6) and repeated use of Lemma 3.3 gives therefore $H^0(G/B, L(E)) = w_1 w_2 \cdots w_N w_0(E) = E$.

Theorem 6.1 reduces the problem of extendibility for B-modules to the semi-simple rank 1 case. To solve the problem in that case note first that if E is a B-module and $G/B = \mathbf{P}^1$ then E extends to G iff the induced bundle $L(E)$ is trivial, i.e. iff $H^*(L(E(-\delta))) = 0$ (see 2.6). This last condition can be transformed into an intrinsic condition on E via the Corollaries 2.3 and 2.4:

THEOREM 6.2. *Let G be a group of semi-simple rank* 1. *A finite dimensional B-module E extends to G iff for all weights χ in E and all $e \in E_\chi$ we have*

if $\langle \alpha^\vee, \chi \rangle \geqslant 1$ there exists $m \geqslant \langle \alpha^\vee, \chi \rangle$ such that $x_{-\alpha,m}e \neq 0$;

if $\langle \alpha^\vee, \chi \rangle \leqslant -1$ there exists $m \geqslant -\langle \alpha^\vee, \chi \rangle$ such that e belongs to the image of $x_{-\alpha,m}$.

This theorem is equivalent to the final result of [6].

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